On The Existence Of Category Bicompletions

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Abstract: *A completeness conjecture is advanced concerning the free small-colimit completion $\mathscr{P}(\mathscr{A})$ of a (possibly large) category $\mathscr{A}$. The conjecture is based on the existence of a small generating-cogenerating set of objects in $\mathscr{A}$. We sketch how the validity of the result would lead to the existence of an Isbell-Lambek bicompletion $\mathscr{C}(\mathscr{A})$ of such an $\mathscr{A}$, without a "change-of-universe" procedure being necessary to describe or discuss the bicompletion*

All categories, functors, and natural transformations, etc., shall be relative to a basic complete and cocomplete symmetric monoidal closed category $\mathscr{V}$ with all intersections of subobjects. A tentative conjecture, based partly on the results of [3], is that if $\mathscr{A}$ is a (large) category containing a small generating and cogenerating set of objects, then $\mathscr{P}(\mathscr{A})$ (which is the free small-colimit completion of $\mathscr{A}$ with respect to $\mathscr{V}$) is not only cocomplete (as is well known), but also complete with all intersections of subobjects.

If this conjecture is true, then one can establish the existence of a resulting "Isbell-Lambek" bicompletion of such an $\mathscr{A}$, along the lines of [1] §4, using the Yoneda embedding $Y : \mathscr{A} \subset \mathscr{P}(\mathscr{A})$. This proposed bicompletion, denoted here by $\mathscr{C}(\mathscr{A})$, has the same "size" as $\mathscr{A}$ and is, roughly speaking, the (replete) closure in $\mathscr{P}(\mathscr{A})$, under both iterated limits and intersections of subobjects, of the class (i.e. large set) of all representable functors from $\mathscr{A}^{\mathrm{op}}$ to $\mathscr{V}$.

More precisely, one can construct $\mathscr{C}(\mathscr{A})$ directly using the Isbell-conjugacy adjunction

$$\begin{tikzcd} \mathscr{P}(\mathscr{A}) \arrow[r, shift left, "Lan_Y(Z)"] & \mathscr{P}(\mathscr{A}^{\mathrm{op}})^{\mathrm{op}} \arrow[l, shift left, "R"] \end{tikzcd}$$

whose existence (see [3] §9) follows from the conjectured completeness of both $\mathscr{P}(\mathscr{A})$ and $\mathscr{P}(\mathscr{A}^{\mathrm{op}})$, and where $Z : \mathscr{A} \to \mathscr{P}(\mathscr{A}^{\mathrm{op}})^{\mathrm{op}}$ is the dual of the Yoneda embedding $\mathscr{A}^{\mathrm{op}} \subset \mathscr{P}(\mathscr{A}^{\mathrm{op}})$. Thus we proceed by factoring the left adjoint $Lan_Y(Z)$ as a reflection followed by a conservative left adjoint

$$\begin{tikzcd} \mathscr{P}(\mathscr{A}) \arrow[r, shift left, "Lan_Y(Z)"] \arrow[d, bend right, "\cup"] & \mathscr{P}(\mathscr{A}^{\mathrm{op}})^{\mathrm{op}} \arrow[l, shift left, "R"] \arrow[dl, shift left] \\ \mathscr{C}(\mathscr{A}) \arrow[ur, shift left, "cons"'] & \end{tikzcd}$$

Such a factorization exists by [1] Theorem 2.1 and is essentially unique by [1] Proposition 5.1. Moreover, the induced full embedding

$$\mathscr{A} \subset \mathscr{C}(\mathscr{A})$$

then preserves any small limit or small colimit that already exists in $\mathscr{A}$.

One important consequence is that various results from [2] on monoidal biclosed completion of categories can be accordingly revamped using such a bicompletion $\mathscr{C}(\mathscr{A})$; see also [3] §7, which describes some examples where $\mathscr{P}(\mathscr{A})$ is monoidal or monoidal biclosed. Note that here especially one could conveniently avoid the awkward "change-of-$\mathscr{V}$-universe" procedure employed in [2].

References.

[1] B. J. Day, "On Adjoint-Functor Factorization", Lecture Notes in Mathematics, 420 (Springer-Verlag 1974), Pg. 1-19.

[2] B. J. Day, "On Closed Categories Of Functors II", Lecture Notes in Mathematics, 420 (Springer-Verlag 1974), Pg. 20-54.

[3] B. J. Day and S. Lack, "Limits Of Small Functors" J. Pure Appl. Alg., 210 (2007), Pg. 651-663.

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